GCE: 503, Analysis and measure theory
May 2017
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Exercise 1:
Let $(X, \mathcal{A}, \mu)$ be a measure space. Let A_n be a sequence in $\mathcal{A}$ such that $\mu(A_n)$ converges to zero.
(i). Prove or disprove:
if $f : X \to [0, \infty)$ is a measurable function and $\mu(X) < \infty$ then $\int_{A_n} f$ converges to zero.
(ii). Let g be in $L^1(X)$. Show that $\int_{A_n} g$ converges to zero.

Exercise 2:
Let (X, d) be a bounded metric space. For any non empty subset S of X and x in X we define:

$$d(x, S) = \inf\{d(x, s) : s \in S\}$$

.
If A and B are two non empty subsets of X we define:

$$d_H(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)\}.$$

(i). If $d_H(A, B) = 0$, are A and B necessarily equal? (prove or disprove).
(ii). Let $\mathcal{C}$ be the set of all non empty closed subsets of X. Show that d_H defines a metric on $\mathcal{C}$.

Exercise 3:
Let $(X, \mathcal{A}, \mu)$ be a measure space and $\{f_k\}$ a sequence in $L^p(X)$ where $1 \leq p \leq \infty$. Suppose that $\{f_k\}$ converges in $L^p(X)$ to f. Show that f_k converges in measure to f on X.
Hint: According to the definition of convergence in measure, you need to show that for any positive ϵ, $\mu(\{x \in X : |f_k(x) - f(x)| \geq \epsilon\})$ converges to zero as k tends to infinity.

Exercise 4:
Suppose $g_n, g \in L^1(\mathbb{R})$, g_n converges to g almost everywhere, and $\int g_n$ converges to $\int g$. Define $f_n(x) := g_n(x + n)$.
(i). Prove or disprove: there exists an f in $L^1(\mathbb{R})$ such that f_n converges to f almost everywhere.
(ii). Prove or disprove: if there is an f as in (i), then $\int f_n$ converges to $\int f$.